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Lost equivalence of nonlinear sigma and CP^1 models on noncommutative space

Hideharu Otsu
Faculty of Economics, Aichi University
Toyohashi, Aichi 441-8522, Japan
E-mail: otsu@vega.aichi-u.ac.jp

Toshiro Sato
Faculty of Policy Science, Matsusaka University
Matsusaka, Mie 515-8511, Japan
E-mail: tsato@matsusaka-u.ac.jp

Hitoshi Ikemori
Faculty of Economics, Shiga University
Hikone, Shiga 522-8522, Japan
E-mail: ikemori@biwako.shiga-u.ac.jp

Shinsaku Kitakado
General Education, Faculty of Science and Technology, Meijo University
Tempaku, Nagoya 486-8502, Japan
E-mail: kitakado@ccmfs.meijo-u.ac.jp

Abstract: We show that the equivalence of nonlinear sigma and CP^1 models which is valid on the commutative space is broken on the noncommutative space. This conclusion is arrived at through investigation of new BPS solitons that do not exist in the commutative limit.

Keywords: Sigma Models, Solitons Monopoles and Instantons, Non-Commutative Geometry.

Contents

1. Introduction

Field theories on the noncommutative space have been extensively studied in the last few years. Particularly, BPS solitons are worth examined, because they might not share the common features with those on the commutative space [1]–[3].

Solitons in the CP^1 model on two dimensional noncommutative space have been studied in [4] and further developed in [5] in connection with the dynamical aspects of the theory. The non-BPS solitons, that do not exist in the commutative case, have been studied in [6]. In the previous paper [7], we have reported on a set of new BPS solitons in the noncommutative CP^1 model, that does not exist in the commutative limit. On the other hand, solitons in the $\mathrm{U}(n)$ nonlinear sigma model have been studied on the noncommutative space [8]–[10].

In this paper, we investigate the new aspects one encounters when the two dimensional space is promoted to the noncommutative space. We consider the nonlinear sigma model, which has been discussed as the modified U(2) sigma model in [8, 9], and CP^1 model defined on the noncommutative space $\mathbb{R}^2_{NC}$ with commutative time. Although our discussions are concerned with the static solitons on $\mathbb{R}^2_{NC} \times \mathbb{R}$, these solutions can also be considered as the instanton solutions on $\mathbb{R}^2_{NC}$.

On the commutative space $\mathbb{R}^2$, there exists a definite correspondence between the configurations of the nonlinear sigma model and the configurations of the CP^1 model and both models are equivalent. We shall see, however, that on the noncommutative space such a correspondence is destroyed. In fact, there exists a BPS soliton in the nonlinear sigma model that does not have the counterpart in the CP^1 model. Furthermore, the correspondence relation between the two models is different for solitons and anti-solitons. A new BPS anti-soliton solution in the CP^1 model has been found on the noncommutative space that does not exist in the commutative space.

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In what follows, after a brief description of the CP^1 model and the nonlinear sigma model on the commutative space in the section 2, we proceed, in section 3, to the description of these models and their BPS solitons on the noncommutative space. In section 4 we investigate the subtle properties of the BPS solitons discussed in section 3. Section 5 is devoted to summary.

2. Models on the commutative space

In this section we shall describe the CP^1 model and the nonlinear sigma model on the commutative space. We focus our attention to the topological properties of the configurations and their interrelations in both models.

The field variable in CP^1 model is given by

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2.1}$$

with the constraint

$$\Phi^\dagger \Phi = 1 \,, \tag{2.2}$$

which implies that Φ spans the S^3 space. The lagrangian is written as

$$L = \int d^2x (|D_t\Phi|^2 - \sum_{i=1}^{2} |D_i\Phi|^2) \,, \tag{2.3}$$

where D_a is a covariant derivative defined as

$$\begin{aligned} D_a\Phi &= \partial_a\Phi - i\Phi A_a \,, \\ A_a &= -i\Phi^\dagger \partial_a \Phi \,. \end{aligned} \tag{2.4}$$

This system has a local U(1) symmetry under the transformation $\Phi \to e^{i\alpha}\Phi$. Consequently, the static configuration is characterized by the homotopy class $\Pi_2(S^3/S^1) = Z$. The corresponding topological charge can be written as

$$\begin{aligned} Q &= \frac{-i}{2\pi} \int d^2x \epsilon_{ij} (D_i\Phi^\dagger D_j\Phi) \\ &= \frac{-i}{2\pi} \int d^2x \epsilon_{ij} (\partial_i\Phi^\dagger \partial_j\Phi) \,. \end{aligned} \tag{2.5}$$

Although, the second equality is valid on the commutative space, it is not the case on the noncommutative space. In section 3, we shall define the topological charge through the first covariant form [4, 11]. We have the following energy bound for the static configuration,

$$E = \int d^2x \sum_{i=1}^{2} |D_i\Phi|^2 \geq 8\pi\, |Q| \,, \tag{2.6}$$

and the equality is satisfied for the BPS soliton (anti-soliton).

Next we recapitulate the notations of the nonlinear sigma model that is in fact equivalent to the CP^1 model on the commutative space. Using the variable n^a with the constraint $\sum_{a=1}^{3}(n^a)^2 = 1$, the lagrangian can be written as

$$L = \int d^2x \left[(\partial_t n^a)^2 - (\partial_i n^a)^2\right], \tag{2.7}$$

and the topological charge is expressed as

$$Q = \frac{1}{8\pi}\int d^2x \epsilon_{ij}\epsilon_{abc} n^a \partial_i n^b \partial_j n^c . \tag{2.8}$$

Relation with the CP^1 variable Φ is

$$n^a = \Phi^\dagger \sigma^a \Phi , \tag{2.9}$$

which leads to the equalities of (2.3) with (2.7) and of (2.5) with (2.8) .

Using the projector $P \equiv \Phi\Phi^\dagger$ $\left(P^2 = P\right)$, we can express the nonlinear sigma model in terms of the variable U

$$U \equiv 2P - 1 \tag{2.10}$$

which satisfies

$$U^2 = 1 . \tag{2.11}$$

Lagrangian, energy of static configuration, topological charge are rewritten as

$$L = \frac{1}{2}\int d^2x \,\mathrm{tr}\left[(\partial_t U)^2 - 2\partial_{\bar z}U\partial_z U\right], \tag{2.12}$$

$$E = \int d^2x \,\mathrm{tr}(\partial_{\bar z}U\partial_z U) \tag{2.13}$$

and

$$Q = \frac{1}{16\pi}\int d^2x \,\mathrm{tr}\left[U\left(\partial_{\bar z}U\partial_z U - \partial_z U\partial_{\bar z}U\right)\right], \tag{2.14}$$

respectively. Here "tr" denotes the trace of 2×2 matrices. The static configurations of nonlinear sigma model satisfy $\sum_{a=1}^{3}(n^a)^2 = 1$ or equivalently $U^2 = 1$. Thus they are classified in terms of the homotopy class $\Pi_2\left(S^2\right) = Z$ and the corresponding topological charges are (2.8) and (2.14) . The expressions using the variables n^a and those using U are equivalent on the commutative space, where we have the relation

$$U = n^a\sigma^a . \tag{2.15}$$

However, this is not the case on the noncommutative space. In section 3 we shall extend the nonlinear sigma model to the noncommutative space using the lagrangian, topological charge written in terms of U .

Finally, we note in passing that the configuration in CP^1 model and the configuration in the nonlinear sigma model can be related with each other (2.9). This relation can also be solved for Φ

$$\Phi = \frac{1}{\sqrt{2}}\frac{e^{i\alpha}}{\sqrt{1-n^3}}\begin{pmatrix} n^1 + in^2 \\ 1 - n^3 \end{pmatrix}. \tag{2.16}$$

Furthermore, if the use is made of the relation (2.16), the lagrangians are also equivalent. Consequently, on the commutative space, the nonlinear sigma model and the CP^1 model are equivalent.

3. Models on the noncommutative space

In this section we shall investigate the CP^1 and the nonlinear sigma models on the noncommutative space $\mathbb{R}^2_{NC}$ [4]–[7] and [12]–[15]. The space coordinates obey the commutation relation

$$[x, y] = i\theta \tag{3.1}$$

or

$$[z, \bar{z}] = \theta > 0\,, \tag{3.2}$$

when written in the complex variables, $z = \frac{1}{\sqrt{2}}(x + iy)$ and $\bar{z} = \frac{1}{\sqrt{2}}(x - iy)$. The Hilbert space can be described in terms of the energy eigenstates $|n\rangle$ of the harmonic oscillator whose creation and annihilation operators are $\bar{z}$ and z respectively,

$$\begin{aligned} z\,|n\rangle &= \sqrt{\theta n}\,|n-1\rangle\,, \\ \bar{z}\,|n\rangle &= \sqrt{\theta(n+1)}\,|n+1\rangle\,. \end{aligned} \tag{3.3}$$

Space integrals on the commutative space are replaced by the trace on the Hilbert space

$$\int d^2x \Rightarrow \mathrm{Tr}_{\mathcal{H}}\,, \tag{3.4}$$

where, $\mathrm{Tr}_{\mathcal{H}}$ denotes the trace over the Hilbert space as

$$\mathrm{Tr}_{\mathcal{H}}\mathcal{O} = 2\pi\theta \sum_{n=0}^{\infty} \langle n|\,\mathcal{O}\,|n\rangle\,. \tag{3.5}$$

The derivatives with respect to z and $\bar{z}$ are defined by $\partial_z = -\theta^{-1}[\bar{z},]$ and $\partial_{\bar{z}} = \theta^{-1}[z,]$.

The CP^1 lagrangian is

$$L = \mathrm{Tr}_{\mathcal{H}}(|D_t\Phi|^2 - |D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2)\,, \tag{3.6}$$

where Φ is a 2-component complex vector with the constraint $\Phi^\dagger\Phi = 1$. The covariant derivative is defined by

$$D_a\Phi = \partial_a\Phi - i\Phi A_a, A_a = -i\Phi^\dagger\partial_a\Phi\,. \tag{3.7}$$

For the static configuration, topological charge and energy are given by

$$Q = \frac{-i}{2\pi}\,\mathrm{Tr}_{\mathcal{H}}(\epsilon_{ij}D_i\Phi^\dagger D_j\Phi) = \frac{1}{2\pi}\,\mathrm{Tr}_{\mathcal{H}}\left(|D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2\right) \tag{3.8}$$

and

$$E = \mathrm{Tr}_{\mathcal{H}}\left(|D_z\Phi|^2 + |D_{\bar{z}}\Phi|^2\right) \geq 2\pi\,|Q|\,. \tag{3.9}$$

The configuration which saturates the energy bound satisfies the BPS soliton equation

$$D_{\bar{z}}\Phi = (1 - \Phi\Phi^\dagger)z\Phi = 0 \tag{3.10}$$

or BPS anti-soliton equation

$$D_z\Phi = (1 - \Phi\Phi^\dagger)\bar{z}\Phi = 0\,. \tag{3.11}$$

The following BPS soliton (anti-soliton) solutions are known. The solutions that have the counterparts in the CP^1 model on the commutative space [4] are soliton solutions

$$W = \begin{pmatrix} z^n \\ 1 \end{pmatrix} \tag{3.12}$$

with $Q = n, E = 2\pi n$ and anti-soliton solutions

$$W = \begin{pmatrix} \bar{z}^n \\ 1 \end{pmatrix} \tag{3.13}$$

with $Q = -n, E = 2\pi n$. Here

$$\Phi = W \frac{1}{\sqrt{W^\dagger W}} \, . \tag{3.14}$$

(3.12) and (3.13) when expressed in terms of $P = \Phi\Phi^\dagger$ are respectively

$$P = \begin{pmatrix} z^n \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & z^n \dfrac{1}{\bar{z}^n z^n + 1} \\ \dfrac{1}{\bar{z}^n z^n + 1} \bar{z}^n & \dfrac{1}{\bar{z}^n z^n + 1} \end{pmatrix} \tag{3.15}$$

and

$$P = \begin{pmatrix} \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} z^n & \bar{z}^n \dfrac{1}{z^n \bar{z}^n + 1} \\ \dfrac{1}{z^n \bar{z}^n + 1} z^n & \dfrac{1}{z^n \bar{z}^n + 1} \end{pmatrix} . \tag{3.16}$$

Furthermore, there are solutions that do not exist on the commutative space [7], they are $Q = n, E = 2\pi n$ soliton solutions;

$$\Phi = \begin{pmatrix} \dfrac{1}{\sqrt{\prod_{k=1}^n (\bar{z}z + k\theta)}} z^n \\ \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix} \tag{3.17}$$

and $Q = -n, E = 2\pi n$ anti-soliton solutions;

$$\Phi = \begin{pmatrix} \bar{z}^n \dfrac{1}{\sqrt{\prod_{k=1}^n (\bar{z}z + k\theta)}} \\ 0 \end{pmatrix} . \tag{3.18}$$

(3.17) and (3.18) when expressed in terms of $P = \Phi\Phi^\dagger$ are respectively

$$P = \begin{pmatrix} 1 & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle \langle m| \end{pmatrix} \tag{3.19}$$

and

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle \langle m| & 0 \\ 0 & 0 \end{pmatrix} . \tag{3.20}$$

Next we turn to the nonlinear sigma model on the noncommutative space. We start from the model on the commutative space expressed in terms of U. Lagrangian and topological charge are respectively

$$\begin{aligned} L &= \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left((\partial_t U)^2 - 2\partial_{\bar z} U \partial_z U\right)\right] \\ &= \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}(\partial_t P)^2\right] - \theta^{-2}\,\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left([z,P][P,\bar z]\right)\right] \end{aligned} \tag{3.21}$$

and

$$Q = \frac{1}{16\pi\theta^2}\,\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left(U\left([\bar z,U][z,U] - [z,U][\bar z,U]\right)\right)\right]. \tag{3.22}$$

Energy for the static configuration is expressed as

$$\begin{aligned} \theta^2 E &= \mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}([z,P]\,[P,\bar z])\right] \\ &= \mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left(P\,[\bar z,P]\,[z,P] - P\,[z,P]\,[\bar z,P]\right)\right] + \mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left(F^\dagger F + F F^\dagger\right)\right] \\ &\geq \mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left(P\,[\bar z,P]\,[z,P] - P\,[z,P]\,[\bar z,P]\right)\right] \\ &= 2\pi\theta^2 Q + \frac{1}{2}\,\mathrm{Tr}_{\mathcal{H}}\left[\mathrm{tr}\left([\bar z,P]\,[z,P] - [z,P]\,[\bar z,P]\right)\right], \end{aligned} \tag{3.23}$$

where $F = (1-P)zP$ [9, 16]. The second term on the last line of eq. (3.23) is zero for the finite energy configuration. Consequently, the energy bound

$$E \geq 2\pi Q \tag{3.24}$$

is satisfied for $Q > 0$. The BPS soliton equation [9, 16, 17] is

$$(1-P)zP = 0\,. \tag{3.25}$$

Similarly, for $Q < 0$ the BPS anti-soliton equation is

$$(1-P)\bar z P = 0\,. \tag{3.26}$$

With $P = \Phi\Phi^\dagger$, the BPS equations (3.25) and (3.26) are consistent with those of CP^1 model.

Finally, let us see the relation of configurations in the noncommutative CP^1 model with those of nonlinear sigma model. The configuration of nonlinear sigma model can be obtained from that of CP^1 model through $U = 2\Phi\Phi^\dagger - 1$ or $P = \Phi\Phi^\dagger$. Obtaining CP^1 configuration from that of nonlinear sigma model is not straightforward. In what follows we shall see a concrete relation between the configurations of both models considering as an example the BPS soliton (anti-soliton) configuration.

The CP^1 BPS soliton (3.12), (3.17) (anti-soliton (3.13), (3.18)) are solutions of the nonlinear sigma model through the relation $P = \Phi\Phi^\dagger$. Consider next a new BPS soliton of the nonlinear sigma model expressed in terms of P,

$$P = \begin{pmatrix} \sum_{m=0}^{k-1} |m\rangle\langle m| & 0 \\ 0 & \sum_{m=0}^{n-1} |m\rangle\langle m| \end{pmatrix}, \tag{3.27}$$

which satisfies the BPS soliton equation (3.25). Topological charge and energy are $Q = k+n$ and $E = 2\pi(k+n)$, respectively. If we require $P = \Phi\Phi^\dagger$ for this configuration, we have

$$\Phi^\dagger\Phi = \sum_{m=0}^{k+n-1} |m\rangle\langle m| \neq 1\,, \tag{3.28}$$

which shows the absence of the corresponding CP^1 configuration. We note, in this connection that Φ can be expressed as

$$\Phi = \begin{pmatrix} \sum_{m=0}^{k-1} |m\rangle\langle m+n| \\ \sum_{m=0}^{n-1} |m\rangle\langle m| \end{pmatrix}, \tag{3.29}$$

but $\Phi^\dagger\Phi = 1$ is not valid for finite k . Only for $k \to \infty$ we have $\Phi^\dagger\Phi = 1$ which leads to the CP^1 soliton (3.17) or (3.19).

Furthermore, consider a BPS anti-soliton

$$P = \begin{pmatrix} 1 - \sum_{m=0}^{n-1} |m\rangle\langle m| & 0 \\ 0 & 1 - \sum_{m=0}^{k-1} |m\rangle\langle m| \end{pmatrix}, \tag{3.30}$$

which satisfies BPS anti-soliton equation (3.26). Topological charge and energy are $Q = -(k+n)$ and $E = 2\pi(k+n)$, respectively. When k is sent to infinity, it reduces to the CP^1 anti-soliton (3.18) or (3.20). If we require $P = \Phi\Phi^\dagger$ for the anti-soliton (3.30), we get for example

$$\Phi = \begin{pmatrix} \sum_{m=n}^{\infty} |m\rangle\langle 2(m-n)+1| \\ \sum_{m=k}^{\infty} |m\rangle\langle 2(m-k)| \end{pmatrix}, \tag{3.31}$$

which satisfies $\Phi^\dagger\Phi = 1$. In this case, although the CP^1 configuration does exist, it does not have the commutative limit due to the fact that infinitely many “dislocations” of the state prevent us from approaching the continuous configuration.

From what we have seen, we may conclude that the correspondence between the configurations of the CP^1 model and the nonlinear sigma model is destroyed. Concretely, we have shown that the soliton which cannot exist in the CP^1 model can be found in the nonlinear sigma model as a BPS soliton. Furthermore, we have found a new BPS anti-soliton solution (3.31) of CP^1 model which does not exist on the commutative space.

4. Properties of soliton solutions

In this section, we shall analyze the properties of the solitons discussed in the previous sections.

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4.1 Anti-solitons in the CP^1 model

First let us look for the general form of CP^1 anti-soliton that corresponds to the anti-soliton (3.30) in the nonlinear sigma model. We can solve

$$\Phi\Phi^\dagger = \begin{pmatrix} 1-\sum_{m=0}^{n-1}|m\rangle\langle m| & 0 \\ 0 & 1-\sum_{m=0}^{k-1}|m\rangle\langle m| \end{pmatrix} \tag{4.1}$$

as

$$\Phi = \begin{pmatrix} \alpha\mu^\dagger \\ \beta\nu^\dagger \end{pmatrix}. \tag{4.2}$$

Here α, β satisfy

$$\alpha^\dagger\alpha = 1\,, \qquad \beta^\dagger\beta = 1 \tag{4.3}$$

and μ, ν

$$\begin{aligned} \mu^\dagger\mu &= 1\,, \qquad \nu^\dagger\nu = 1\,, \\ \mu^\dagger\nu &= 0\,, \qquad \nu^\dagger\mu = 0\,, \end{aligned} \tag{4.4}$$

$$\mu\mu^\dagger + \nu\nu^\dagger = 1\,, \tag{4.5}$$

and consequently

$$\Phi^\dagger\Phi = 1\,, \tag{4.6}$$

thus we can confirm that Φ is a variable of CP^1 model. Using this Φ we have

$$P = \Phi\Phi^\dagger = \begin{pmatrix} \alpha\alpha^\dagger & 0 \\ 0 & \beta\beta^\dagger \end{pmatrix}. \tag{4.7}$$

Let us introduce the operators S_N and P_N ,

$$S_N \equiv \sum_{m=0}^{\infty}|m+N\rangle\langle m|\,, \qquad P_N \equiv \sum_{m=0}^{N-1}|m\rangle\langle m|\,, \tag{4.8}$$

with the properties

$$\begin{aligned} S_N^\dagger S_N &= 1\,, \qquad S_N S_N^\dagger = 1 - P_N\,, \\ P_N S_N &= 0 = S_N^\dagger P_N\,. \end{aligned} \tag{4.9}$$

We can express as

$$\alpha = S_N\,, \qquad \beta = S_K\,, \tag{4.10}$$

and write P as

$$P = \begin{pmatrix} S_N S_N^\dagger & 0 \\ 0 & S_K S_K^\dagger \end{pmatrix} = \begin{pmatrix} 1-P_N & 0 \\ 0 & 1-P_K \end{pmatrix}, \tag{4.11}$$

thus P satisfies the anti-BPS equation (3.26).

If we choose μ, ν as

$$\mu = \sum_{p=0}^{\infty}|2p+1\rangle\langle p|\,, \qquad \nu = \sum_{p=0}^{\infty}|2p\rangle\langle p|\,, \tag{4.12}$$

we can show that

$$\mu^\dagger\mu = \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} |p'\rangle \langle 2p'+1|\, 2p+1\rangle \langle p| = \sum_{p=0}^{\infty} |p\rangle \langle p| = 1\,,$$
$$\nu^\dagger\nu = \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} |p'\rangle \langle 2p'|\, 2p\rangle \langle p| = \sum_{p=0}^{\infty} |p\rangle \langle p| = 1\,,$$
$$\mu\mu^\dagger = \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} |2p'+1\rangle \langle p'|\, p\rangle \langle 2p+1| = \sum_{p=0}^{\infty} |2p+1\rangle \langle 2p+1|\,,$$
$$\nu\nu^\dagger = \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} |2p'\rangle \langle p'|\, p\rangle \langle 2p| = \sum_{p=0}^{\infty} |2p\rangle \langle 2p|\,,$$
$$\nu^\dagger\mu = \sum_{p'=0}^{\infty}\sum_{p=0}^{\infty} |p'\rangle \langle 2p'|\, 2p+1\rangle \langle p| = 0\,, \tag{4.13}$$

thus satisfying (4.4) and (4.5). This gives

$$\Phi = \begin{pmatrix} S_n\mu^\dagger \\ S_k\nu^\dagger \end{pmatrix} = \begin{pmatrix} \sum\limits_{m=n}^{\infty} |m\rangle \langle 2(m-n)+1| \\ \sum\limits_{m=k}^{\infty} |m\rangle \langle 2(m-k)| \end{pmatrix} \tag{4.14}$$

which appeared in the previous section (3.31) .

It is interesting to note that S_N and P_N can be used to express our solutions found in the previous work [7] for anti-soliton (3.18) and (3.20)

$$\Phi = \begin{pmatrix} S_n \\ 0 \end{pmatrix}, \qquad P = \begin{pmatrix} S_n S_n^\dagger & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1-P_n & 0 \\ 0 & 0 \end{pmatrix} \tag{4.15}$$

and for soliton (3.17) and (3.19)

$$\Phi = \begin{pmatrix} S_n^\dagger \\ P_n \end{pmatrix}, \qquad P = \begin{pmatrix} S_n^\dagger S_n & S_n^\dagger P_n \\ P_n S_n & P_n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & P_n \end{pmatrix}. \tag{4.16}$$

These expressions are in accord with the solution generating technique appearing in [18]–[20].

4.2 Classification of noncommutative solitons

We shall show in this subsection that the values of $\operatorname{tr} P$ at the boundary of the Hilbert space can be used in classifying the solitons discussed in section 3. The trace of field variable P at the boundary of the Hilbert space is defined as

$$\langle \operatorname{tr} P\rangle_\infty \equiv \lim_{n\to\infty} \langle n| \operatorname{tr} P\, |n\rangle\,. \tag{4.17}$$

The general configurations P must be a vacuum at the boundary of the Hilbert space and, as we shall see, $\langle \operatorname{tr} P\rangle_\infty$ takes the values $0, 1, 2$. It can be easily verified that for the solutions (3.15), (3.16), (3.19) and (3.20) $\langle \operatorname{tr} P\rangle_\infty = 1$, for (3.27) $\langle \operatorname{tr} P\rangle_\infty = 0$, and for (3.30) $\langle \operatorname{tr} P\rangle_\infty = 2$.

The typical example of the vacuum configuration with zero energy,

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.18}$$

corresponds to $\langle \mathrm{tr}\, P \rangle_\infty = 1.$[1]. The vacua with $\langle \mathrm{tr}\, P \rangle_\infty = 0, 2$ are respectively

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.19}$$

and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.20}$$

The solitons can be considered to be the excited states from the respective vacua. On the other hand, for the nonlinear sigma model on the commutative space, we have

$$P = \frac{1}{2} + \frac{1}{2}\sum_{a=1}^{3} n^a \sigma^a, \tag{4.21}$$

where the space dependence is in n^a. Consequently, on the commutative space, we always have

$$\langle \mathrm{tr}\, P \rangle_\infty \Rightarrow \lim_{|x|\to\infty} \mathrm{tr}\, P = 1, \tag{4.22}$$

thus the configurations with $\langle \mathrm{tr}\, P \rangle_\infty = 0, 2$ are characteristic of the nonlinear sigma model on the noncommutative space. As we shall see, $\langle \mathrm{tr}\, P \rangle_\infty$ is a conserved quantity against the continuous deformation of configuration with finite energy. Consequently, configurations of the nonlinear sigma model are classified in terms of the topological charge Q and the value of $\langle \mathrm{tr}\, P \rangle_\infty$. In what follows, we shall show that $\langle \mathrm{tr}\, P \rangle_\infty$ is conserved under the continuous deformations of the configurations.

We consider first the vacuum configurations with energy $E = 0$. Let us parametrize P as

$$P = \begin{pmatrix} a & b \\ b^\dagger & c \end{pmatrix}. \tag{4.23}$$

From

$$P^\dagger = P, \tag{4.24}$$

we have

$$a^\dagger = a, \qquad c^\dagger = c. \tag{4.25}$$

If we define

$$A \equiv [a, \bar{z}], \quad B \equiv [b, \bar{z}], \quad C \equiv \left[b^\dagger, \bar{z}\right], \quad D \equiv [c, \bar{z}], \tag{4.26}$$

[1]We shall see that the general vacuum configurations with $\langle \mathrm{tr}\, P \rangle_\infty = 1$ are (4.37) and (4.38).

the energy of the static configuration can be written as

$$
\begin{aligned}
E &= \frac{1}{\theta^2}\,\mathrm{Tr}_{\mathcal{H}}\left\{\mathrm{tr}([z,P]\,[P,\bar{z}])\right\} \\
&= \frac{2\pi}{\theta}\sum_{n=0}^{\infty}\langle n|\left(A^\dagger A + B^\dagger B + C^\dagger C + D^\dagger D\right)|n\rangle \\
&= 0\,.
\end{aligned}
\tag{4.27}
$$

Consequently, for all $|n\rangle$ we have

$$
\langle n|\left(A^\dagger A + B^\dagger B + C^\dagger C + D^\dagger D\right)|n\rangle = 0\,,
\tag{4.28}
$$

from which it follows that

$$
A = B = C = D = 0\,.
\tag{4.29}
$$

Thus taking into account (4.25), each component of P is constant for the configuration with $E = 0$, and P can be written as

$$
P = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix},
\tag{4.30}
$$

where a, c are real numbers and b is complex. Next from

$$
P^2 = P\,,
\tag{4.31}
$$

we have

$$
\begin{aligned}
a^2 + b\bar{b} &= a\,, \\
ab + bc &= b\,, \\
\bar{b}b + c^2 &= c\,.
\end{aligned}
\tag{4.32}
$$

Solving these we are lead to the two conditions given by

$$
a = \lambda_\pm\,, \qquad c = \lambda_\pm\,,
\tag{4.33}
$$

where

$$
\lambda_\pm \equiv \frac{1 \pm \sqrt{1 - 4\,|b|^2}}{2}\,,
\tag{4.34}
$$

and

$$
a + c = 1 \qquad \text{for} \qquad b \neq 0\,.
\tag{4.35}
$$

Possible vacuum configurations are classified into the following three types. First, the configurations connected to

$$
P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},
\tag{4.36}
$$

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can be parametrized by the complex number b ($0 \leq |b| \leq \frac{1}{2}$) as

$$P = \begin{pmatrix} \lambda_+ & b \\ \bar{b} & \lambda_- \end{pmatrix} \tag{4.37}$$

and

$$P = \begin{pmatrix} \lambda_- & b \\ \bar{b} & \lambda_+ \end{pmatrix}. \tag{4.38}$$

When we continuously deform the configuration in the region $0 \leq |b| \leq \frac{1}{2}$, $\langle \mathrm{tr}\, P \rangle_\infty$ remains 1. The other two types of vacuum configurations are

$$P = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.39}$$

with $\langle \mathrm{tr}\, P \rangle_\infty = 0$ and

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{4.40}$$

with $\langle \mathrm{tr}\, P \rangle_\infty = 2$. These two vacuum configurations cannot be deformed keeping $E = 0$ under the conditions (4.34) and (4.35). Consequently, the vacua with different values of $\langle \mathrm{tr}\, P \rangle_\infty$ are disconnected against the continuous deformations. Thus, $\langle \mathrm{tr}\, P \rangle_\infty$ is a conserved quantity taking the values $0, 1, 2$ under the continuous deformation keeping $E = 0$.

We consider next the continuous deformation of the general configuration with finite energy. In order to keep the energy

$$E = \frac{2\pi}{\theta} \sum_{n=0}^{\infty} \langle n| \, \mathrm{tr} \left([z, P] \, [P, \bar{z}] \right) |n\rangle \,, \tag{4.41}$$

finite for the static configuration P, the condition,

$$\lim_{n \to \infty} \langle n| \, \mathrm{tr} \left([z, P] \, [P, \bar{z}] \right) |n\rangle = 0 \,, \tag{4.42}$$

is needed as the boundary condition. Consequently, general configuration P must be a vacuum at the boundary of the Hilbert space ($|n\rangle$ with $n \to \infty$), and thus $\langle \mathrm{tr}\, P \rangle_\infty$ takes the value 0 or 1 or 2. As a result, $\langle \mathrm{tr}\, P \rangle_\infty$ is a conserved quantity and the configurations are classified by the topological charge $Q = 0, \pm 1, \pm 2, \ldots$ and $\langle \mathrm{tr}\, P \rangle_\infty = 0, 1, 2$.

5. Summary

On the commutative space, there exists a definite correspondence between the configurations of the nonlinear sigma model and the configurations of CP^1 and both models are equivalent. We have seen, however, that on the noncommutative space such a correspondence is destroyed. In fact, there exist the BPS solitons (3.27) in the nonlinear sigma model that do not have the counterpart in CP^1 model. On the other hand, the new BPS anti-soliton solutions in the CP^1 model have been found in the noncommutative space ((3.31), general form is (4.2)) that do not exist in the commutative space.

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We found that the configurations in the nonlinear sigma model is to be classified not only by the topological charge $Q = 0, \pm 1, \pm 2, \dots$ but also by $\langle \mathrm{tr}\, P \rangle_\infty = 0, 1, 2$. We have seen in section 3, that for the configuration with $\langle \mathrm{tr}\, P \rangle_\infty = 1$ there exist both solitons and anti-solitons. In the case of $\langle \mathrm{tr}\, P \rangle_\infty = 0$, only solitons ($Q > 0$) can exist, while in the case of $\langle \mathrm{tr}\, P \rangle_\infty = 2$ only anti-solitons ($Q < 0$) are confirmed. This asymmetry deserves a further study.

Relations with the gauge theories and use of the solitons in the actual physical problems are interesting topics to be investigated.

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